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Cross Helicity and Turbulent Magnetic Diffusivity in the Solar Convection Zone

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Abstract In a density-stratified turbulent medium the cross helicity $\langle \mathbf{u}' \cdot \mathbf{B}' \rangle$ is considered as a result of the interaction of the velocity fluctuations and a large-scale magnetic field. By means of a quasilinear theory and by numerical simulations we find the cross helicity and the mean vertical magnetic field anti-correlated. In the high-conductivity limit the ratio of the helicity and the mean magnetic field equals the ratio of the magnetic eddy diffusivity and the (known) density scale height. The result can be used to predict that the cross helicity at the solar surface exceeds the value of 1 gauss $\mathrm{km\,s^{-1}}$. Its sign is anti-correlated with that of the radial mean magnetic field. Alternatively, we can use our result to determine the value of the turbulent magnetic diffusivity from observations of the cross helicity.

Keywords: Sun: magnetic field – Magnetohydrodynamics (MHD)

1. Introduction

Dynamo theory for convective zones needs to know both the values of the α-effect and the eddy diffusivity. The α-effect is strongly related to the kinetic helicity which has opposite signs in the two hemispheres. Almost all of the theoretical calculations for rotating stratified turbulence lead to negative helicity (*i.e.* positive α-effect) for the northern hemisphere and positive helicity (*i.e.* negative α-effect) for the southern hemisphere. Despite all of the complications to measure the helicity on the solar surface, a new result has recently been presented by Komm, Hill, and Howe (2008). They do indeed find negative (positive) values for the

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kinetic helicity in the northern (southern) hemisphere. This result is based on a ring-diagram analysis of GONG data. The values remained constant as long as the magnetic field did not exceed 10 gauss (G). Using the observation of the DIV-CURL correlation, $\mathcal{C} = \langle (u_{,x} + v_{,y})(v_{,x} - u_{,y}) \rangle$, which is proportional to the kinetic helicity (Rüdiger, Brandenburg, and Pipin, 1999; see Rüdiger and Hollerbach (2004) for more details), Duvall and Gizon (2000) found $\mathcal{C}$ to be negative (positive) in the northern (southern) hemisphere, as derived from the horizontal velocity components of mesogranulation patterns. Egorov, Rüdiger, and Ziegler (2004) simulated these observations with the NIRVANA code and reproduced them with Taylor numbers as small as 10^3.

By use of the finding of Keinigs (1983) the current helicity and the α-effect are anti-correlated, so one also can derive the sign of the α-effect by observation of the current helicity $\langle \mathbf{J}' \cdot \mathbf{B}' \rangle$. Seehafer (1990) started to observe the current helicity at the solar surface showing that it is negative (positive) in the northern (southern) hemisphere. Again the α-effect is found to be positive (negative) in the northern (southern) hemisphere.

The numerical value of the helicity derived by Komm *et al.* (2008) is of the order of 10^{-7} cm s^{-2}. This turns out to be very small, because the resulting α-effect is below 1 cm s^{-1}. By comparison, Käpylä, Korpi, and Brandenburg (2009) find $\alpha \simeq 0.03 u_{\rm rms}$ near the surface from their convection simulations. With $u_{\rm rms} \simeq 300$ m s^{-1}, this corresponds to 10 m s^{-1}. The maximal α-value in their box center is of the order of $0.3 u_{\rm rms}$. This highlights a major discrepancy between theory and observations or, at least, a difficulty in determining α from observations.

The empirical definition of the turbulent magnetic diffusion seems to be more straightforward. The decay of non-permanent magnetic structures such as sunspots or larger active regions lead to numerical values of the turbulent magnetic diffusivity. One finds $\eta_{\rm T} \simeq 10^{11}$ cm^2 s^{-1} from sunspot decay (Stix, 1989) or $\eta_{\rm T} \simeq 10^{12}$ cm^2 s^{-1} from the decay of active regions (Schrijver and Martin, 1990). These values are smaller than the value of 3×10^{12} cm^2 s^{-1}, which results from the widely used formula $\eta_{\rm T} \approx u_{\rm rms} \ell / 3$ with correlation or mixing length ℓ and parameter values taken close to the surface. There is no possibility until now to observe the turbulent diffusivity on the solar surface for the quiet Sun where the magnetic quenching of this quantity by magnetic fields is negligible. We shall demonstrate in the present paper that there is a rather simple possibility to observe the magnetic diffusivity even in the presence of very weak magnetic fields (< 10 G), for which quenching should be negligible.

We shall show that in the presence of a mean magnetic field along the direction of density stratification, hydromagnetic turbulence will attain cross helicity, $\langle \mathbf{u}' \cdot \mathbf{B}' \rangle$, whose value is proportional to the turbulent magnetic diffusivity. Indeed, our work is an extension of that by Kleeorin *et al.* (2003), who considered the effect of stratification of turbulent intensity. More recently, Kuzanyan, Pipin, and Zhang (2007) emphasized the importance of cross-helicity for estimating internal solar parameters important for the dynamo. We mention in this connection that cross helicity itself may constitute a potentially important dynamo effect (Yoshizawa, 1990; Yokoi, 1996). In this paper we propose a measurement of cross-helicity in the Sun for estimating the turbulent magnetic diffusivity in

quiet regions. We argue that this is more accurate than measuring, for example, the mean electromotive force.

2. Mean-Field Electrodynamics

Let $\mathbf{u}' = \mathbf{u} - \langle \mathbf{u} \rangle$ and $\mathbf{B}' = \mathbf{B} - \langle \mathbf{B} \rangle$ be the fluctuations of velocity and magnetic field about an average value denoted by angular brackets. The mean-field dynamo theory of cosmic magnetic fields is based on the relation

$$\langle \mathbf{u}' \times \mathbf{B}' \rangle = \alpha \langle \mathbf{B} \rangle - \beta \langle \mathbf{J} \rangle \tag{1}$$

between the turbulent electromotive force $\mathcal{E} = \langle \mathbf{u}' \times \mathbf{B}' \rangle$ and the mean-field quantities $\langle \mathbf{B} \rangle$ and $\langle \mathbf{J} \rangle$, where $\mathbf{J}$ is the mean current density. For the purpose of this discussion we neglect here possible effects of mean flows on the correlators, *i.e.* we assume $\langle \mathbf{u} \rangle = 0$. Note the basic difference between the quantities α and β in that β is a scalar while α is a pseudoscalar. For rotating stars a pseudoscalar can be formed by use of the basic rotation rate, *e.g.*, $\alpha \propto (\mathbf{g} \cdot \mathbf{\Omega})$ with gravity $\mathbf{g}$ as the only remaining preferred direction apart from $\mathbf{\Omega}$. Hence, the amplitude of the α-effect must mainly be influenced by the Coriolis number

$$\Omega^* = 2\tau_{\rm corr}\Omega, \tag{2}$$

where $\tau_{\rm corr}$ is the correlation time of the dominating mode of turbulence. However, Ω^* is very small at the solar surface so that the α-effect in Equation (1) cannot be observed directly.

The parameter β in Equation (1) exists even in nonrotating plasmas. It is thus not governed by the Coriolis number Ω^* and is therefore not small by comparison. It is, however, not possible to observe by direct means the mean current density $\langle \mathbf{J} \rangle$ at the solar surface. The decay of sunspots should provide good estimates for β when the induction equation is solved by using Equation (1); and the time-dependent solutions are compared with the observations (Krause and Rüdiger, 1975). Though successful, this procedure cannot serve as a proof of the existence of Equation (1). We must conclude, therefore, that the basic Equation (1) cannot be tested with observations taken from the solar surface. This is an unsatisfying situation given that Equation (1) is a fundamental relation of a whole branch of cosmic MHD and, of course, there is no better laboratory than the Sun to probe such basic relations.

Fortunately, the situation is quite different for another correlation between fluctuations of flow and field, namely the cross helicity $\langle \mathbf{u}' \cdot \mathbf{B}' \rangle$, which itself is a pseudoscalar. It is straightforward to formulate the relation

$$\langle \mathbf{u}' \cdot \mathbf{B}' \rangle = \alpha_{\rm c} \langle \mathbf{g} \cdot \mathbf{B} \rangle - \beta_{\rm c} \langle \mathbf{\Omega} \cdot \mathbf{J} \rangle \tag{3}$$

similar to Equation (1). The $\alpha_{\rm c}$-effect does *not* run with the Coriolis number Ω^*. Similar to the α-effect in Equation (1) the $\alpha_{\rm c}$ in Equation (3) is of the dimension of a velocity but this velocity should be much larger than the corresponding α in Equation (1). As the second term on the RHS of Equation (3) only exists in the presence of rotation, it will be negligibly small at the solar surface.

In summary, by simple reasons the observations of the cross correlation $\langle \mathbf{u}' \cdot \mathbf{B}' \rangle$ at the solar surface should give a realistic chance to confirm the existence of relations that are typical for mean-field electrodynamics.

3. Nonconservation of Cross-Helicity in Turbulent Fluids

The cross-helicity is conserved globally (as volume integral) in ideal incompressible fluids (Woltjer, 1958). In view of the conservation law, it may be anticipated that the balance of small-scale cross-helicity should be treated globally by defining the small-scale cross-helicity sources, cross-helicity fluxes, and formulating the dynamical cross-helicity equation similar to the approach used to study the balance of magnetic helicity. Examples of such an approach to the cross-helicity problem can be found in the literature (Sur and Brandenburg, 2009). In this section we show, however, that cross-helicity is not conserved in turbulent fluids such as the solar convection zone and its balance is controlled by local processes.

The turbulence is known to dissipate efficiently the quantities, which are conserved in ideal fluids (with zero diffusivities). The well known example is the energy balance. Kinetic energy is conserved in ideal hydrodynamics. The rate of energy dissipation in Kolmogorov (1941) picture of turbulence is, however, constant independent of whatever small (but finite) is the viscosity. The same is true about almost all quantities conserved in ideal fluids. Turbulent fragmentation of scales cascades rapidly the quantities to the smallest scales where they dissipate.

The only known exception is magnetic helicity that is conserved even in turbulent fluids. The reason can be seen from the following. In the simplest case of isotropic homogeneous turbulence, the spectrum tensor of fluctuating magnetic fields can be written as

$$B_{ij}(\mathbf{k}) = \frac{E^{\mathrm{m}}(k)}{8\pi k^2}\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right) - \frac{\mathrm{i}H^{\mathrm{m}}(k)}{8\pi k^2}\varepsilon_{ijn}k_n, \tag{4}$$

where E^{m} and H^{m} are magnetic energy and helicity spectra,

$$\langle B'^2 \rangle = \int\limits_0^\infty E^{\mathrm{m}}(k)\mathrm{d}k, \quad \langle \mathbf{B}' \cdot \mathbf{A}' \rangle = \int\limits_0^\infty H^{\mathrm{m}}(k)\mathrm{d}k, \quad \mathbf{B}' = \mathrm{rot}\mathbf{A}'. \tag{5}$$

The spectrum tensor (4) is positive definite, $B_{ij}C_iC_j^* \geq 0$ (Bochner, 1933), where $\mathbf{C}$ is an arbitrary vector and the asterisk marks complex conjugation. For the tensor (4), this leads to the inequality (Moffatt, 1969)

$$|H^{\mathrm{m}}(k)|k \leq E^{\mathrm{m}}(k), \tag{6}$$

which is also known as the realizability condition. Imagine that at some (small) wavenumber k_1 helicity is finite and the spectral magnetic helicity to energy ratio at that wavenumber is $k_0^{-1} = |H^{\mathrm{m}}(k_1)|/E^{\mathrm{m}}(k_1)$. If helicity could follow magnetic energy in its cascade to large k, the ratio $H^{\mathrm{m}}/E^{\mathrm{m}}$ would be constant across the

spectrum. Then, Equation (6) would require $k \leq k_0$, which is an inequality that is impossible to satisfy for spectra with sufficiently broad inertial range. Therefore, magnetic helicity cannot be cascaded to diffusive scales to dissipate. The helicity is conserved, and the conservation law has important consequences for large-scale dynamos (Brandenburg and Subramanian, 2005).

The conservation of magnetic helicity is, however, an exception. For example, for *kinetic* helicity, which is also conserved in ideal hydrodynamics, instead of inequality (6) we have $|H^{\rm k}| \leq E^{\rm k} k$ with no restrictions for the kinetic helicity cascade to viscous scales. As a consequence, kinetic helicity is not conserved. The same is true for the cross helicity. This can also be seen by comparing the rates of helicity dissipation. Using the fact that vorticity and current density scale inversely proportional to the square roots of viscosity and magnetic diffusivity, respectively, we see that the rate of magnetic helicity dissipation decreases with decreasing magnetic diffusivity proportional to its square root, while that of cross helicity is independent of viscosity and magnetic diffusivity and does therefore not vanish. The cross-helicity balance is controlled by local processes. In spite of some striking similarities in the saturation of dynamos controlled by magnetic and cross helicity, the presence of significant cross-helicity dissipation as well as the forcing term in the momentum equation destroy the nice analogy (Sur and Brandenburg, 2009). In the following we proceed with deriving the cross-helicity from local relations.

4. Quasilinear Theory of Cross Helicity

In this section we derive the symmetric part $\langle u'_i B'_j \rangle^{\rm s} = (\langle u'_i B'_j \rangle + \langle u'_j B'_i \rangle)/2$ of the cross correlation tensor $\langle u'_i B'_j \rangle$. The pseudotensor $\langle u'_i B'_j \rangle$ can be finite only in the presence of a mean magnetic field $\langle \mathbf{B} \rangle$ and for inhomogeneous fluids. The required inhomogeneity can be due to stratification of density or turbulent intensity as well as the inhomogeneity of the mean field itself.

The turbulent flow is assumed anelastic, so that $\mathrm{div}(\rho \mathbf{u}') = 0$. It is convenient to use the Fourier transformation of the momentum density $\mathbf{m} = \rho \mathbf{u}'$, *i.e.*

$$\mathbf{m}(\mathbf{r}, t) = \int \hat{\mathbf{m}}(\mathbf{k}, \omega)\ \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{r} - \omega t)} \mathrm{d}\mathbf{k}\ \mathrm{d}\omega, \tag{7}$$

and similarly for the fluctuation of the magnetic field. The linearized equation for magnetic fluctuations in terms of the Fourier amplitudes reads

$$\begin{aligned}(-\mathrm{i}\omega \ &+\ \eta k^2)\hat{B}'_i(\mathbf{k}, \omega) = \\ &= \ \mathrm{i}k_j \int \Bigg(\hat{m}_i(\mathbf{k} - \mathbf{k}', \omega - \omega') \left(\frac{\hat{B}_j}{\rho}\right)(\mathbf{k}', \omega') - \\ &- \ \hat{m}_j(\mathbf{k} - \mathbf{k}', \omega - \omega') \left(\frac{\hat{B}_i}{\rho}\right)(\mathbf{k}', \omega') \Bigg)\ \mathrm{d}\mathbf{k}' \mathrm{d}\omega',\end{aligned} \tag{8}$$

where $\hat{\mathbf{B}}$ is the Fourier transform of the mean magnetic field.

The spectral tensor of the momentum density that accounts for the stratification of the turbulence to first order terms reads

$$
\begin{aligned}
\langle \hat{m}_i(\mathbf{z},\omega)\hat{m}_j(\mathbf{z}',\omega')\rangle = \delta(\omega+\omega')\frac{\hat{q}(k,\omega,\kappa)}{16\pi k^2} \ \times \\
\times \left(\delta_{ij} - k_i k_j/k^2 + (\kappa_i k_j - \kappa_j k_i)/(2k^2)\right),
\end{aligned}
\tag{9}
$$

where $\mathbf{k} = (\mathbf{z}-\mathbf{z}')/2,\ \boldsymbol{\kappa} = \mathbf{z}+\mathbf{z}'$, $\hat{q}$ is the Fourier transform of the local spectrum,

$$
q(k,\omega,\mathbf{r}) = \rho^2 E(k,\omega,\mathbf{r}) = \int \hat{q}(k,\omega,\boldsymbol{\kappa})\mathrm{e}^{\mathrm{i}\boldsymbol{\kappa}\cdot\mathbf{r}}\ \mathrm{d}\boldsymbol{\kappa}, \tag{10}
$$

so that

$$
\langle u'^2\rangle = \int_0^\infty\int_0^\infty E(k,\omega,\mathbf{r})\ \mathrm{d}k\,\mathrm{d}\omega. \tag{11}
$$

Derivation of the cross correlation yields

$$
\begin{aligned}
\langle u_i' B_j'\rangle^{\mathrm{s}} &= \frac{1}{2}\eta_{\mathrm{T}}\left(G_i\langle B_j\rangle + G_j\langle B_i\rangle\right) + \left(\frac{1}{10}\eta_{\mathrm{T}} + \frac{4}{15}\hat{\eta}\right)\delta_{ij}\left(\mathbf{U}\cdot\langle\mathbf{B}\rangle\right) + \\
&+ \left(\frac{1}{10}\eta_{\mathrm{T}} - \frac{1}{15}\hat{\eta}\right)\left(U_i\langle B_j\rangle + U_j\langle B_i\rangle\right) \\
&- \left(\frac{3}{10}\eta_{\mathrm{T}} + \frac{2}{15}\hat{\eta}\right)\left(\langle B_{j,i}\rangle + \langle B_{i,j}\rangle\right),
\end{aligned}
\tag{12}
$$

where $\mathbf{G} = \nabla\log\rho$ and $\mathbf{U} = \nabla\log\langle u'^2\rangle$ are the gradients of density and turbulent intensity and

$$
\eta_{\mathrm{T}} = \frac{1}{3}\int_0^\infty\int_0^\infty \frac{\eta k^2 E}{\omega^2+\eta^2 k^4}\mathrm{d}k\,\mathrm{d}\omega, \tag{13}
$$

$$
\hat{\eta} = \int_0^\infty\int_0^\infty \frac{\eta k^2\omega^2 E}{(\omega^2+\eta^2 k^4)^2}\mathrm{d}k\,\mathrm{d}\omega. \tag{14}
$$

Here $\eta = 1/\mu_0\sigma$ is the molecular magnetic diffusivity. Both quantities run with the magnetic Reynolds number for low conductivity ($\sigma \to 0$) and become finite for high conductivity ($\eta \to 0$). From the cross correlation tensor (12) the cross helicity

$$
\langle \mathbf{u}'\cdot\mathbf{B}'\rangle = \eta_{\mathrm{T}}\left(\mathbf{G}\cdot\langle\mathbf{B}\rangle\right) + \left(\frac{\eta_{\mathrm{T}}}{2} + \frac{2\hat{\eta}}{3}\right)\left(\mathbf{U}\cdot\langle\mathbf{B}\rangle\right) \tag{15}
$$

is obtained.

Current observations only supply the correlation $\langle u_r' B_r'\rangle$. From Equation (12) we find

$$
\langle u_r' B_r'\rangle = \eta_{\mathrm{T}} G\langle B_r\rangle - \left(\frac{3\eta_{\mathrm{T}}}{10} + \frac{2\hat{\eta}}{15}\right)\left(2\frac{\partial\langle B_r\rangle}{\partial r} - U\langle B_r\rangle\right), \tag{16}
$$

where $G = G_r$ and $U = U_r$ are the only non-zero radial components of the stratification vectors. Further simplifications can be obtained by using the mixing-length approximation for the turbulence spectrum,

$$E(k, \omega, \mathbf{r}) = 2\langle u'^2 \rangle \delta \left(k - \ell^{-1}\right) \delta(\omega), \quad \eta = \ell^2/\tau_{\rm corr} \tag{17}$$

(Kitchatinov, 1991), where ℓ is mixing length and $\tau_{\rm corr}$ is the correlation time. It yields

$$\langle u'_r B'_r \rangle = \eta_{\rm T} \left(G\langle B_r \rangle - \frac{3}{5}\frac{\partial \langle B_r \rangle}{\partial r} + \frac{3}{10} U\langle B_r \rangle \right). \tag{18}$$

The result can be explained as follows. A rising fluid element $u'_r > 0$ expands so that B'_r has the opposite sign as $\langle B_r \rangle$. The fluid particles which go down, $u'_r < 0$, compress and B'_r has the same sign as B_r. The sign of the product $u'_r B'_r$ is opposite to $\langle B_r \rangle$ in both cases – in accord with the first term on the right hand side (RHS) of Equation (18); note the negativity of G. An upward divergence of the mean field reduces the effect of density stratification. This is realized by the second term on the RHS of Equation (18). The third term shows that also the non-uniformity of the turbulent intensity makes a contribution. However, the contribution of density stratification is dominant, because the density gradient in the upper convection zone is larger than the turbulent intensity gradient. This is already clear from Figure 1 of Krivodubskii and Schultz (1993), who plot, for a solar structure model, the relative contributions from $\mathbf{G}$ and $\mathbf{U}$ in the expression for the usual α effect, where both enter in equal amounts. Here, however, $\mathbf{U}$ enters with a 3/10 factor and is even more subdominant. Thus, we conclude that a finite cross correlation (18) indicates the presence of a large-scale radial field of the opposite sign.

The leading term on the RHS of Equation (18) is due to the density gradient. The resulting relation then reads

$$\frac{\langle u'_r B'_r \rangle}{\langle B_r \rangle} = -\frac{\eta_{\rm T}}{H_\rho}. \tag{19}$$

The magnetic eddy diffusivity can thus be determined if the LHS of Equation (19) is observed and the density scale height H_ρ is known from models of the solar atmosphere.

5. Numerical Simulation

It is straightforward to verify the validity of Equation (19) using numerical simulations of isothermally stratified forced turbulence in a layer with constant gravity, $\mathbf{g} = (0, 0, -g)$ in Cartesian coordinates. In that case the scale height, $H_\rho = c_{\rm s}^2/g$, is constant.

We perform simulations in a cubic domain of size L^3, so the minimal wavenumber is $k \equiv k_1 = 2\pi/L$. We solve the governing equations of compressible magnetohydrodynamics with an isothermal equation of state. The flow is driven by a random forcing function consisting of non-helical waves with wavenumbers

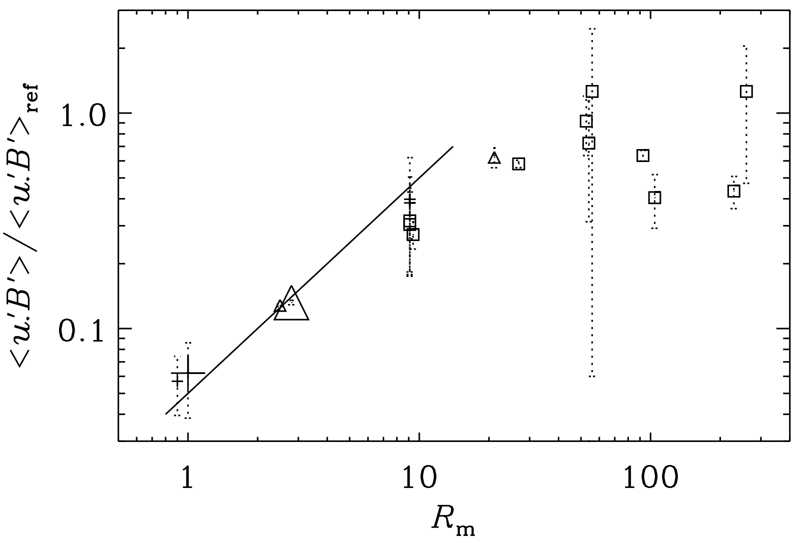

Figure 1. Dependence of the normalized cross helicity on $R_{\rm m}$ for various field strength $B_z/B_{\rm eq} < 0.1$, $P_{\rm m} = 1$, $k_{\rm f}/k_1 = 2.2$, and $H_\rho k_1 = 2.5$. The straight line denotes the fit $\langle \mathbf{u}' \cdot \mathbf{B}' \rangle / \langle \mathbf{u}' \cdot \mathbf{B}' \rangle_{\rm ref} = 0.05 R_{\rm m}$.

whose modulus lies in a narrow band around an average wavenumber $k_{\rm f}$ (which corresponds to ℓ^{-1} used in the previous section). We arrange the amplitude of the forcing function such that the RMS Mach number is around 0.1 or less, so the effects of compressibility are negligible.

In all our runs we adopt stress-free pseudo-vacuum boundary conditions on the top and bottom boundaries, *i.e.* the horizontal magnetic field vanishes. The magnetic field is expressed in terms of the vector potential $\mathbf{A}$ as $\mathbf{B} = \mathbf{B}_0 + \nabla \times \mathbf{A}$, where $\mathbf{B}_0 = (0, 0, B_{0z}) = \text{const}$ is the imposed vertical field which is fixed for each run. The simulations were performed with the Pencil Code[1], which uses sixth-order explicit finite differences in space and third-order accurate time stepping method (Brandenburg and Dobler, 2002). A numerical resolution of up to 256^3 meshpoints was used, depending on the value of the magnetic Reynolds number.

We perform simulations for a number of different parameter combinations. The parameters that are being varied include the strength of the imposed vertical field B_z, the forcing wavenumber $k_{\rm f}$, the gravitational acceleration g, and hence H_ρ, and the values of the magnetic diffusivity. We express these quantities in non-dimensional form and define the magnetic Reynolds number as

$$R_{\rm m} = \frac{u_{\rm rms}}{\eta k_{\rm f}}. \tag{20}$$

[1] `http://pencil-code.googlecode.com`

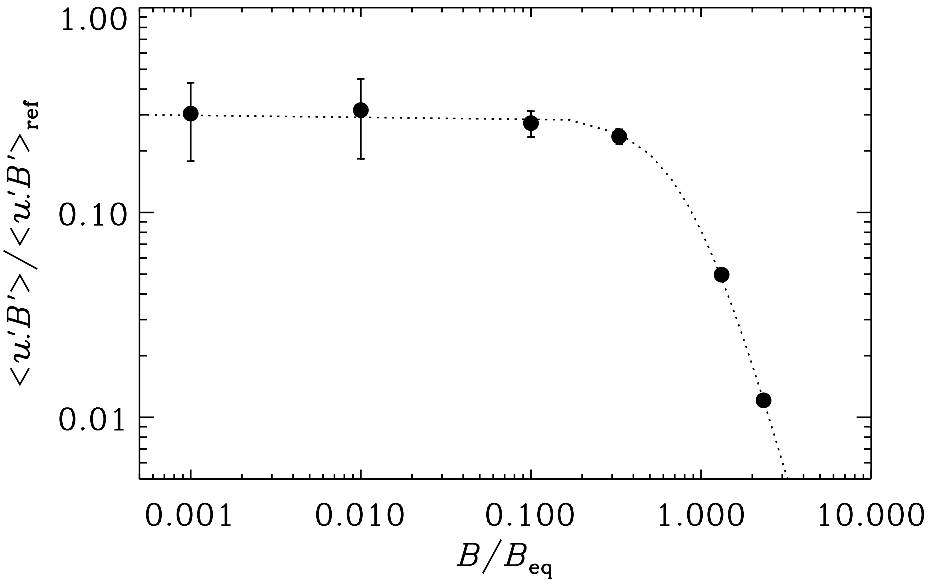

Figure 2. Dependence of the normalized cross helicity on the normalized vertical field strength for $R_\mathrm{m} = 10$, $P_\mathrm{m} = 1$, $k_\mathrm{f}/k_1 = 2.2$, and $H_\rho k_1 = 2.5$. The dotted line corresponds to the graph of $0.3/[1 + (\langle\mathbf{B}\rangle/B_\mathrm{ref})^2]^{3/2}$ with $B_\mathrm{ref} = 0.85\, B_\mathrm{eq}$.

The strength of the magnetic field is characterized by the mean equipartition field strength,

$$B_\mathrm{eq} = \sqrt{\mu_0\langle\rho\rangle}\, u_\mathrm{rms}, \tag{21}$$

which is of the order of 1000 G at the solar surface. We determine the cross helicity, $\langle\mathbf{u}'\cdot\mathbf{B}'\rangle$, as a volume average. In order to relate this to Equation (19) we also need to estimate the value of the turbulent magnetic diffusivity. Earlier work showed that, to a good approximation, η_T can be estimated by Sur, Brandenburg, and Subramanian (2008),

$$\eta_\mathrm{T} \approx \eta_\mathrm{T0} \equiv u_\mathrm{rms}/3k_\mathrm{f}, \tag{22}$$

provided $R_\mathrm{m} \gg 1$, *i.e.* in the high-conductivity approximation. In a number of cases we have verified the validity of this approximation also for the stratified runs shown here.

We present the value of $\langle\mathbf{u}'\cdot\mathbf{B}'\rangle$ in non-dimensional form by dividing by a reference value defined after Equation (19) as

$$\langle\mathbf{u}'\cdot\mathbf{B}'\rangle_\mathrm{ref} = -\frac{\eta_\mathrm{T0}B_0}{H_\rho}. \tag{23}$$

For small R_m the normalized cross helicity depends on R_m (see Figure 1) but it reaches unity for large R_m. It is the expected behavior as Equation (22) gives only a good approximation for Equation (13) for the case of high conductivity,

i.e. for $\sigma \to \infty$. In the opposite case of $\sigma \to 0$, expression (13) vanishes so that the small numbers of the lower-left corner of Figure 1, become understandable. For small values of $R_{\rm m}$ we have $\eta_{\rm T} \propto R_{\rm m}$. For the largest values of $R_{\rm m}$ the error bars for the numerical results are larger. This is mainly because those simulations require larger numerical resolution and long run times become prohibitive.

Figure 2 shows the dependence of the normalized cross helicity, defined by the ratio $\langle \mathbf{u}' \cdot \mathbf{B}' \rangle / \langle \mathbf{u}' \cdot \mathbf{B}' \rangle_{\rm ref}$, on the normalized field strength $B_{0z}/B_{\rm eq}$. Note that the cross helicity is quenched by nearly a factor of 10 for $B_{0z} \approx B_{\rm eq}$.

6. Conclusions

We have shown that nonrotating turbulence at the top of the solar convection zone under the influence of a vertical magnetic field yields a finite cross helicity. The only requirement is the existence of density stratification which enters the induction equation via the anelastic relation $\mathrm{div}(\rho \mathbf{u}') = 0$. The Boussinesq approximation cannot be used. The effect exists mainly in the high-conductivity limit, *i.e.* for sufficiently large magnetic Reynolds numbers (see Figure 1). The radial magnetic field, on the other hand, must be weak enough to remain passive so that it does not dominate the flow. Figure 2 shows that the maximum field is given by $B_{\rm eq}$ which is *much* higher than the mean vertical field, which is of the order of a few gauss on the Sun.

To estimate the value of the cross helicity at the solar surface we shall assume a density scale height of 100 km. Then one finds from Equation (19) that

$$\langle u_r' B_r' \rangle \simeq -\frac{\langle B_r \rangle}{1\mathrm{G}} \frac{\eta_{12}}{H_7} \ \mathrm{G\ km\ s^{-1}}. \tag{24}$$

The average is here to be applied over many turbulent cells which, in the Sun, might correspond to 30–100 Mm. The magnetic diffusivity in Equation (24) has been used in the form $\eta_{\rm T} = 10^{12} \eta_{12}\,\mathrm{cm^2\,s^{-1}}$ and the density scale height as $H_\rho = 100 H_7$ km. We thus predict the existence of a cross helicity of more than $1\ \mathrm{G\,km\,s^{-1}}$. We also emphasize that the cross helicity is anti-correlated to the mean radial magnetic field, *i.e.*

$$\langle u_r' B_r' \rangle \ \langle B_r \rangle < 0. \tag{25}$$

For a dipolar background field the sign of the cross helicity will be opposite in the two hemispheres.

Relation (19) can also be used to measure the magnetic diffusivity if the cross helicity is known by observations. In order to find the cross helicity one only has to correlate the observed flow fluctuations with observed magnetic fluctuations. Together with the calculated mean value of the radial magnetic field, Equation (19) provides the unknown quantity $\eta_{\rm T}$. We hope that such an analysis of the observations using, for example, data from the Hinode satellite will soon provide supporting evidence for an anti-correlation between $\langle u_r' B_r' \rangle$ and $\langle B_r \rangle$, and that a meaningful value of $\eta_{\rm T}$ can be obtained in that way.

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